Ring Theory 7

17 February 2024 17:42

Q> X is a non-empty subset of a comm. ring R.

Annihilator of $X = \{r \in R \mid rx = 0 \ \forall x \in X\}$

Show that X is a subset of $Ann_R(Ann_R(X))$.

Ans:- $Ann_R(Ann_R(X)) = \{r' \in R \mid r'r = 0 \ \forall r \in Ann_R(X)\}$

$\exists r$ st $rx = 0 \ \forall x \in X$

$\exists r'$ st. $r'r = 0 \Rightarrow r' \in X$ because $\forall r \in Ann_R(X)$ $rx = 0$

$\forall x \in X$ we have $rx = 0 \ \forall r \in Ann_R(X)$

Q> $Ann_R(X) = Ann_R(Ann_R(Ann_R(X)))$

Ans:- $Ann_R(Ann_R(Ann_R(X))) \subset Ann_R(X)$

$r \in Ann_R(X)$

$\hookrightarrow rx = 0 \ \forall x \in X$ $\xrightarrow[\text{go to}]{\text{need to}} \{r''\} = \{r\}$

$\hookrightarrow rx = 0 \ \forall x \in \{Ann_R(Ann_R(X)) \setminus Y\}$

$Ann_R(Ann_R(X)) = X \sqcup Y$ (let)

$Ann_R(A) = Ann_R(B)$

$\Rightarrow$

<u>Def</u> :-

Let A be any subset of ring R.

- Then (A) denotes the smallest ideal of R containing A.
  It is called the ideal generated by A.
- RA denote the set of all finite sums of elements of the form ra with $r \in R$ and $a \in A$.
  $RA = \{ r_1 a_1 + \cdots + r_n a_n \mid r_i \in R, a_i \in A, n \in \mathbb{Z}^+ \}$

  $RAR = \{ r_1 a_1 r_1' + \cdots + r_n a_n r_n' \mid r_i, r_i' \in R, a_i \in A, n \in \mathbb{Z}^+ \}$

- An ideal generated by a single element is called the <u>principal ideal</u>.

- An ideal generated by a finite set is called a finitely <u>generated ideal</u>.

'a finitely generated ideal.'

$$(A) = \bigcap_{A \subseteq I} I \quad \cdots [I \text{ is an ideal}]$$

If R is comm. then $RA = AR = RAR = (A)$

Proposition :- Let I be an ideal of R.

1> $I = R$ iff I contains a unit

2> If R is comm. then R is a field iff its only ideal are 0 and R

Proof :- 1> $I = R \Rightarrow 1 \in I$

$u \in U$ and $u \in I \Rightarrow uu^{-1} = 1 \in I$

Then, for $r \in R$ we get,

$r = r \cdot 1 = r(uu^{-1}) = r(u^{-1}u) = (ru^{-1})u = r'u \in I$

$\Rightarrow R \subset I \Rightarrow R = I.$

2> Let R is a field.

Let I be a ideal of R. which is non-zero.

$a \in I, a \neq 0 \Rightarrow aa^{-1} = 1 \Rightarrow 1 \in I$

$\Rightarrow r1 = r \; \forall r \in R \Rightarrow r \in I \; \forall r \in R$

$\Rightarrow I = R \Rightarrow I = 0$ or R

Let, $I = 0$ & R are its only ideal.

Let $r \in R$ be any non-zero element.

$(r) = I = R$

$r \in I$ and $r \in R \Rightarrow rr^{-1} = 1 \in (r)$

$\Rightarrow r^{-1} \in R$

$\Rightarrow R$ is a field.

•> If R is a field then any non-zero ring homomorphism from R to another ring is an injection.

Proof:- $\phi : R \longrightarrow R'$

$\Rightarrow R = I$ or $I = 0$

$\Rightarrow I = 0$ as non-zero homomorphism.

$\Rightarrow \mathrm{Ker}(\phi) = 0$

$\Rightarrow$ Injection

Definition:- An ideal M in an ring S is called a maximal ideal if $M \neq S$ and the only ideals containing M are M and S.

Q> Let A be a commutative ring with 1 and $x \in A$. Prove that the ideal $(x) = xA = A$ iff x is a unit of A.

Ans:- Let, ideal $(x) = xA = A$, then, $x \in xA$

$\Rightarrow x = xy$ for some $y \in A$

As, $xA = A \Longrightarrow 1 = xy$ for some $y \in A$

$\Longrightarrow x$ is a unit

Let x be a unit of A, then,

ideal$(x) = A = xA$

Q> Let $A = \mathbb{Z}[x]$ and let $I = (x, x^2+1)$. Prove that $I = A = \mathbb{Z}[x]$

Ans:- $x^2+1 - xx = 1 \in I \Rightarrow I = A = \mathbb{Z}[x]$